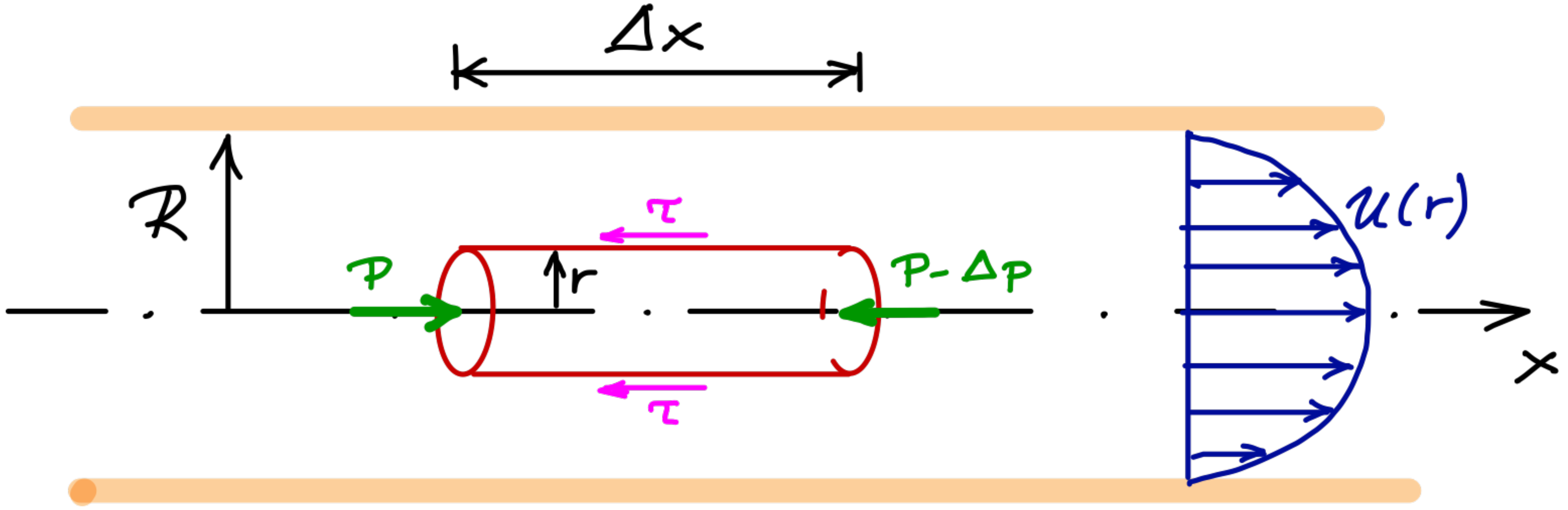
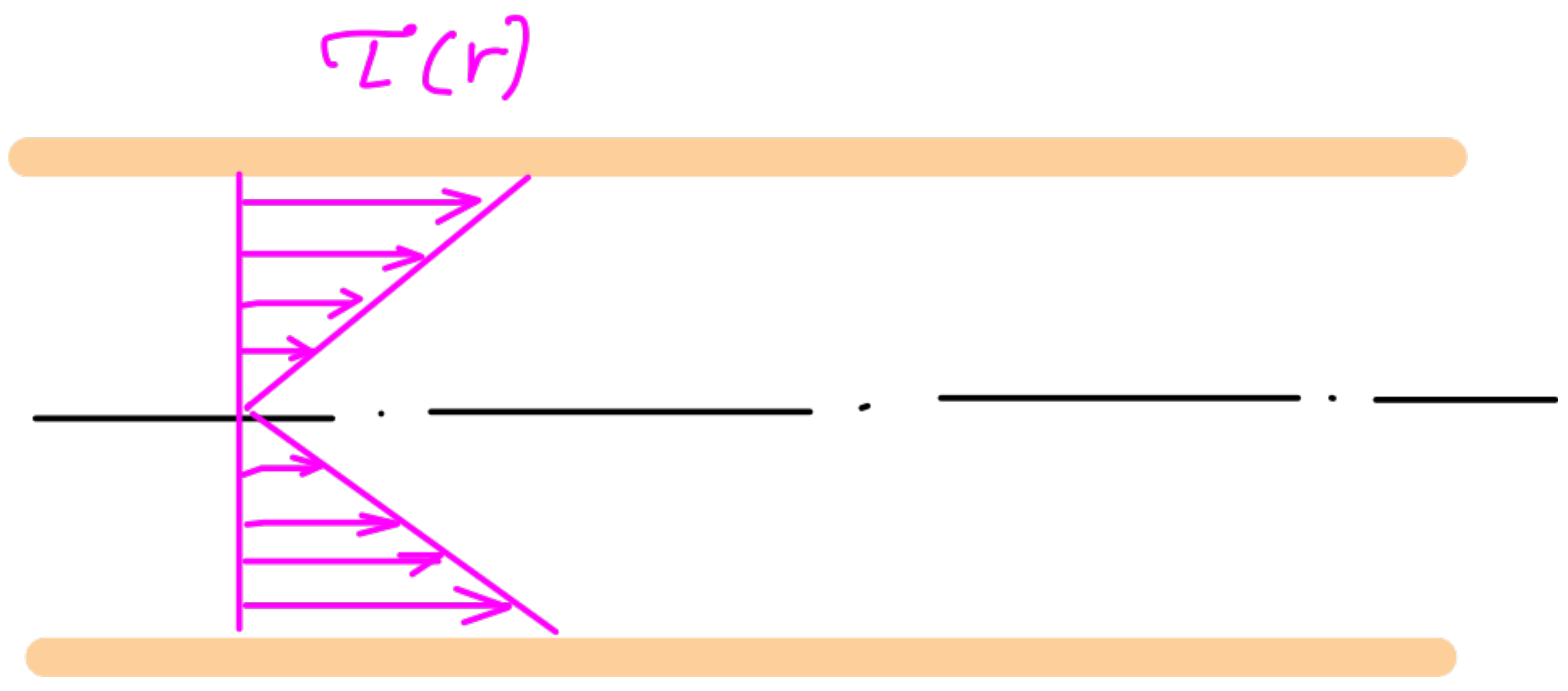
Poiseuille's law





Newton's 2nd law: $\Sigma F_x = 0$

$$\Rightarrow \quad P \cdot \pi r^2 - (P - \Delta p)\pi r^2 - 2\pi r \cdot \Delta x \cdot \tau = 0$$

$$\Rightarrow \quad \Delta p \cdot \pi r^2 = 2\pi r \cdot \Delta x \cdot \tau$$

$$\Rightarrow \quad \boxed{\tau = \frac{\Delta p}{\Delta x} \frac{r}{2}}$$

Maximum shear stres at the wall:

$$\boxed{\tau_w = \frac{\Delta p}{\Delta x} \frac{R}{2}}$$

Relation of pressure drop to velocity and flow (Hagen-Poiseuille law)

Assume Newtonian fluid: $\tau = -\mu \frac{\partial u}{\partial r}$

$$\tau = \frac{\Delta p}{\Delta x}\frac{r}{2} \Rightarrow -\mu \frac{\partial u}{\partial r} = \frac{\Delta p}{\Delta x}\frac{r}{2}$$

$$\Rightarrow \frac{\partial u}{\partial r} = -\frac{1}{2\mu}\frac{\Delta p}{\Delta x} r \Rightarrow \partial u = -\frac{1}{2\mu}\frac{\Delta p}{\Delta x} r\,\partial r$$

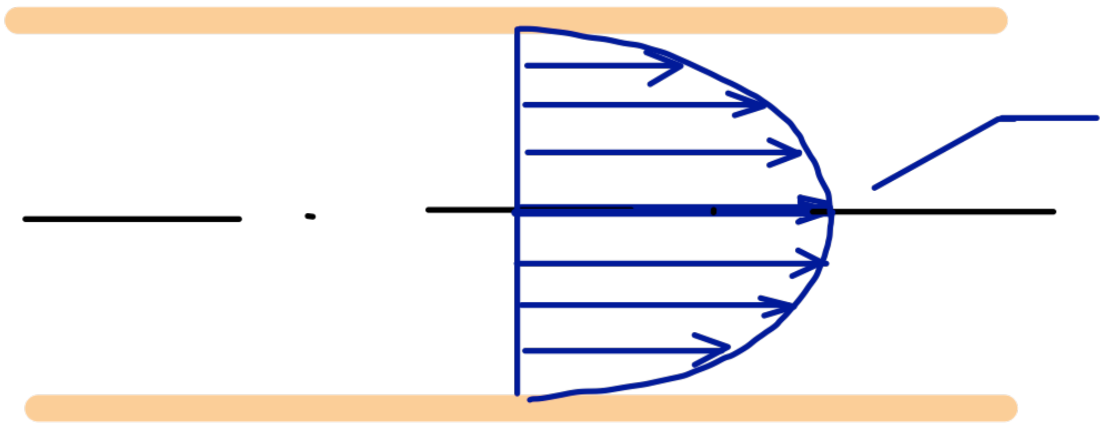
$$\Rightarrow u = -\frac{1}{4\mu}\frac{\Delta p}{\Delta x} r^2 + C_1$$

Boundary condition: $u = 0$ for $r = R$

$$\therefore \quad 0 = -\frac{1}{4\mu}\frac{\Delta p}{\Delta x} R^2 + C_1 \Rightarrow C_1 = \frac{1}{4\mu}\frac{\Delta p}{\Delta x} R^2$$

Hence, $$\boxed{u(r) = \frac{1}{4\mu}\frac{\Delta p}{\Delta x}\left(R^2 - r^2\right)}$$

Parabolic profile



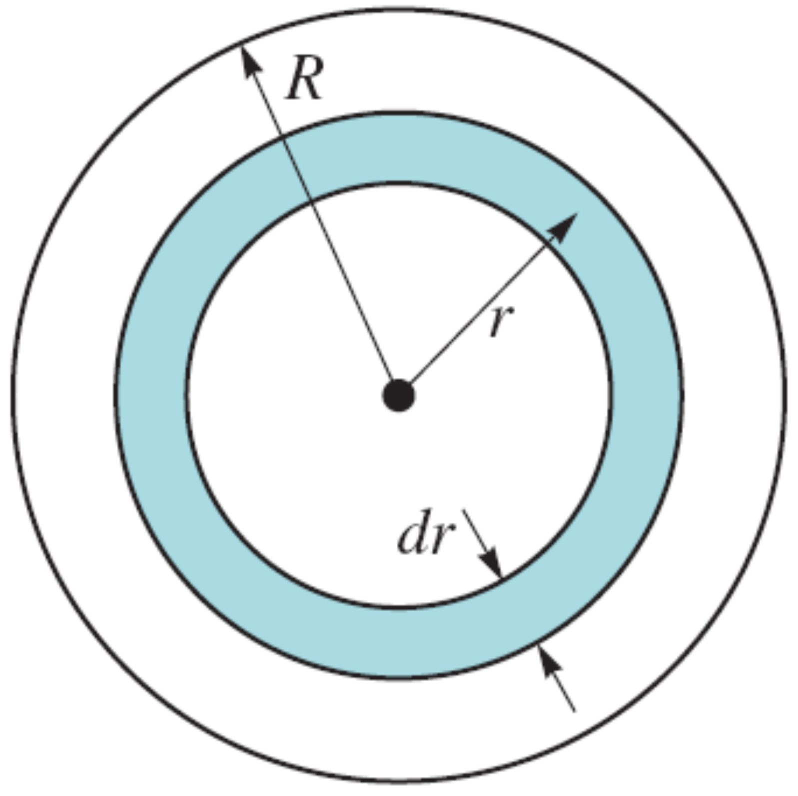
maximum Velocity

V_c @ $r = 0$ (center)

$$V_c = \frac{1}{4\mu}\frac{\Delta p}{\Delta x} R^2$$

$$u(r) = V_c \cdot \left[1 - \frac{r^2}{R^2}\right]$$

Relation of pressure drop to velocity and flow (Hagen-Poiseuille law)



$dA = 2\pi r\, dr$

To get flow, Q, integrate over the cross section

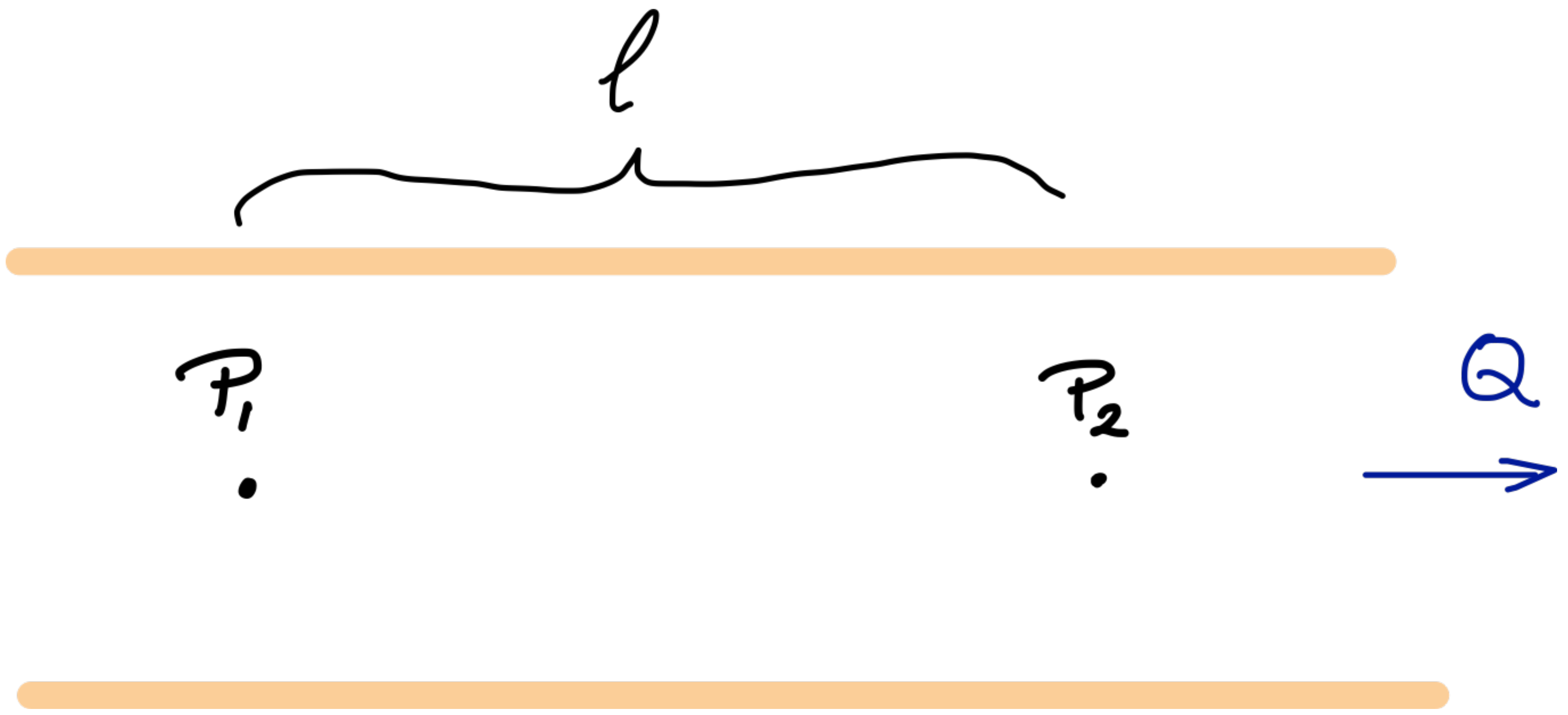
$$Q = \int_0^R u(r)\, 2\pi r\, dr = 2\pi V_c \int_0^R r\left[1 - \frac{r^2}{R^2}\right] dr$$

$$\Rightarrow \quad Q = 2\pi V_c \int_0^R \left[r - \frac{r^3}{R^2}\right] dr = 2\pi V_c \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R$$

$$Q = \frac{\pi R^2 V_c}{2} = \frac{\pi R^4}{8\mu} \frac{\Delta p}{\Delta x}$$

Average velocity: $\bar{V} = \frac{Q}{\pi R^2} = \frac{V_c}{2}$

Relation of pressure drop to velocity and flow (Hagen-Poiseuille law)



Note: $\frac{\Delta p}{\Delta x} = \frac{\Delta p}{\ell} = -\frac{\partial p}{\partial x}$

$\frac{\Delta p}{\Delta x} = \frac{\Delta p}{\ell}$: pressure drop per unit length

$-\frac{\partial p}{\partial x}$: pressure gradient

$$Q = \frac{\pi R^4}{8 \mu \ell} \Delta p$$

or

$$\Delta p = \frac{8 \mu \ell}{\pi R^4} Q$$

By analogy to Ohm's law:

$$\Delta p = \underbrace{\frac{8 \mu \ell}{\pi R^4}}_{\text{Resistance}} Q$$

$V \rightarrow P$

$i \rightarrow Q$

$R \rightarrow R$

Wall shear stress for Poiseuille flow

$$\frac{\Delta p}{\ell} = \frac{2\tau_w}{R} \Rightarrow \tau_w = \frac{\Delta p}{\ell}\frac{R}{2} = \left(\frac{8\mu}{\pi R^4} Q\right)\frac{R}{2} \Rightarrow \boxed{\tau_w = \frac{4\mu}{\pi R^3} Q}$$

Example: Calculate the wall shear stress in the human abdominal aorta ($D = 12\,mm$; $\bar{Q} = 56\ ml/s$) and in external iliac artery ($D = 6.1\ mm$; $\bar{Q} = 6.8\ ml/s$).

Abdominal aorta:

$$\tau_w = \frac{4 \cdot \mu}{\pi R^3} Q = \frac{4 \cdot 0.004\ N.s/m^2}{\pi \cdot 0.006^3\ m^3} \cdot 5.6 \times 10^{-5}\ m^3/s$$

$$\Rightarrow \tau_w = 1.32\ N/m^2$$

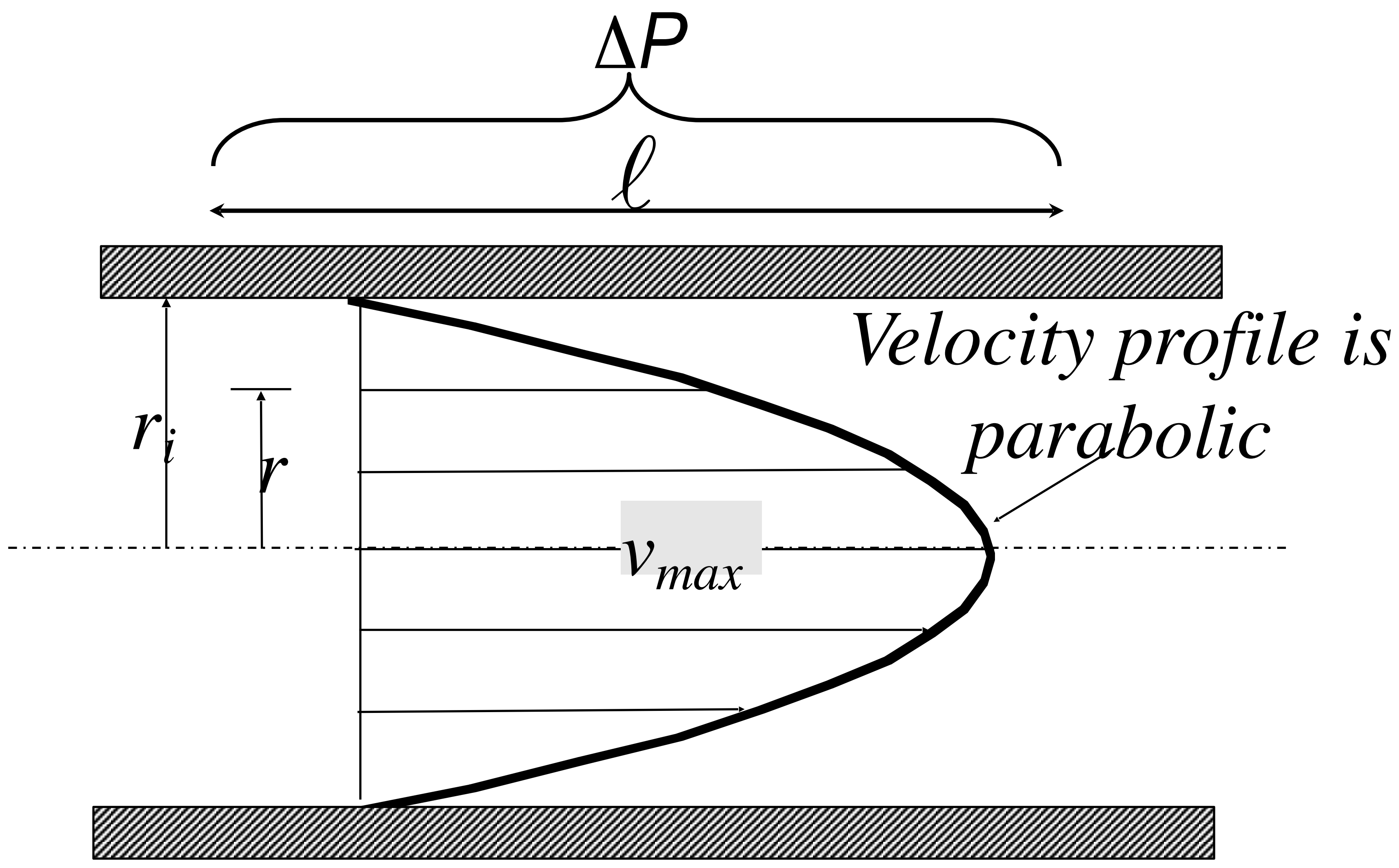
External iliac:

$$\tau_w = \frac{4 \cdot \mu}{\pi R^3} Q = \frac{4 \cdot 0.004\ N.s/m^2}{\pi \cdot 0.00305^3\ m^3} \cdot 6.8 \times 10^{-6}\ m^3/s$$

$$\Rightarrow \tau_w = 1.23\ N/m^2$$

Remark: In general, despite large variations in flow and diameter, τ_w is fairly constant across the arterial tree, with values in the range of $1.0\ N/m^2$ to $1.5\ N/m^2$

Poiseuille's law: summary and limitations



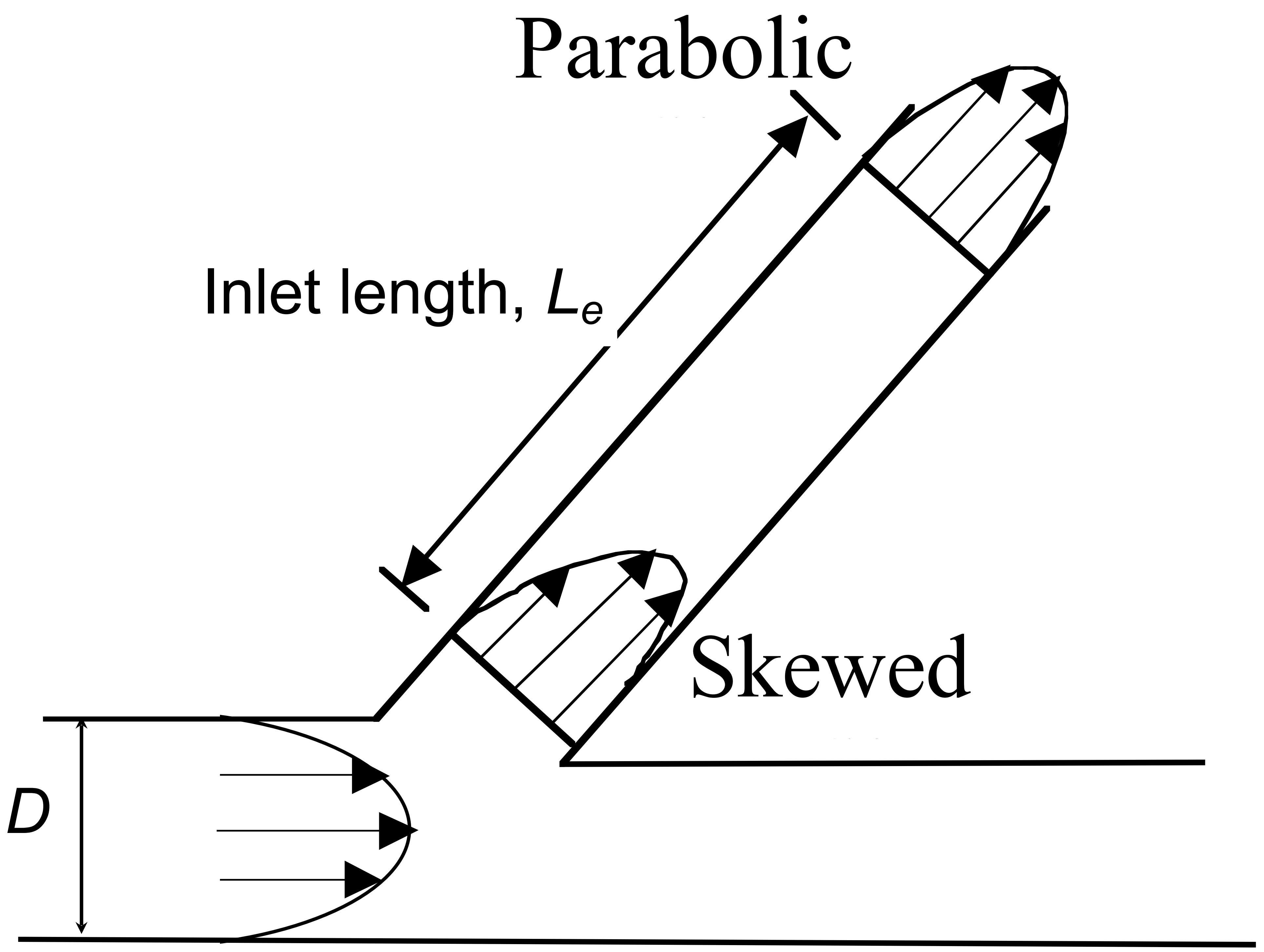
$$v(r) = \frac{1}{4\mu}\frac{\Delta P}{\ell}(r_i^2 - r^2)$$

$$Q = \frac{\pi}{8\mu}\frac{\Delta P}{\ell} r_i^4$$

$$\tau = \frac{4\mu}{\pi r_i^3} Q$$

Conditions:
- Laminar, steady and developed flow
- Newtonian fluid
- Straight, uniform, constant diameter tube
- Rigid wall

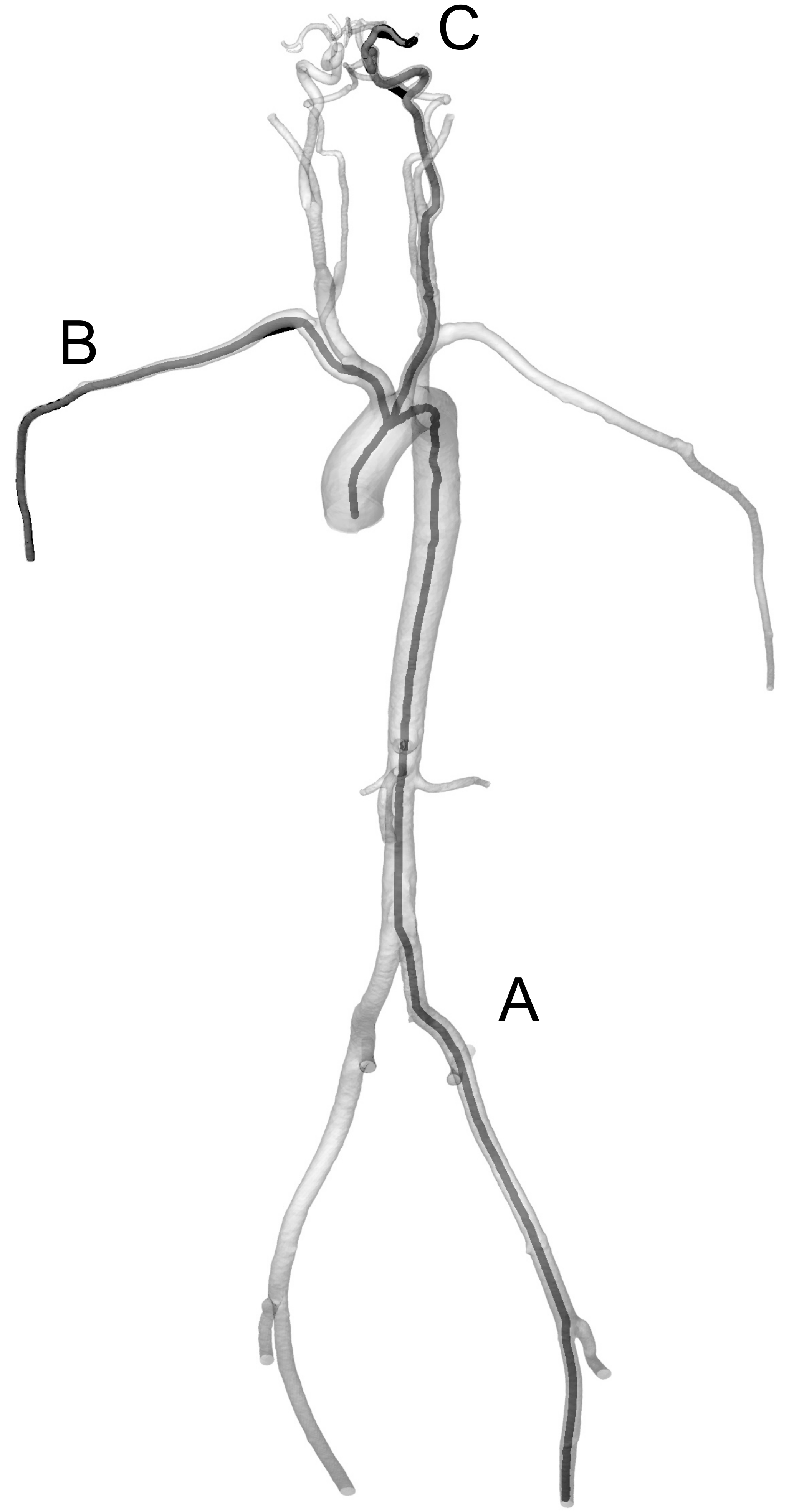
Inlet length

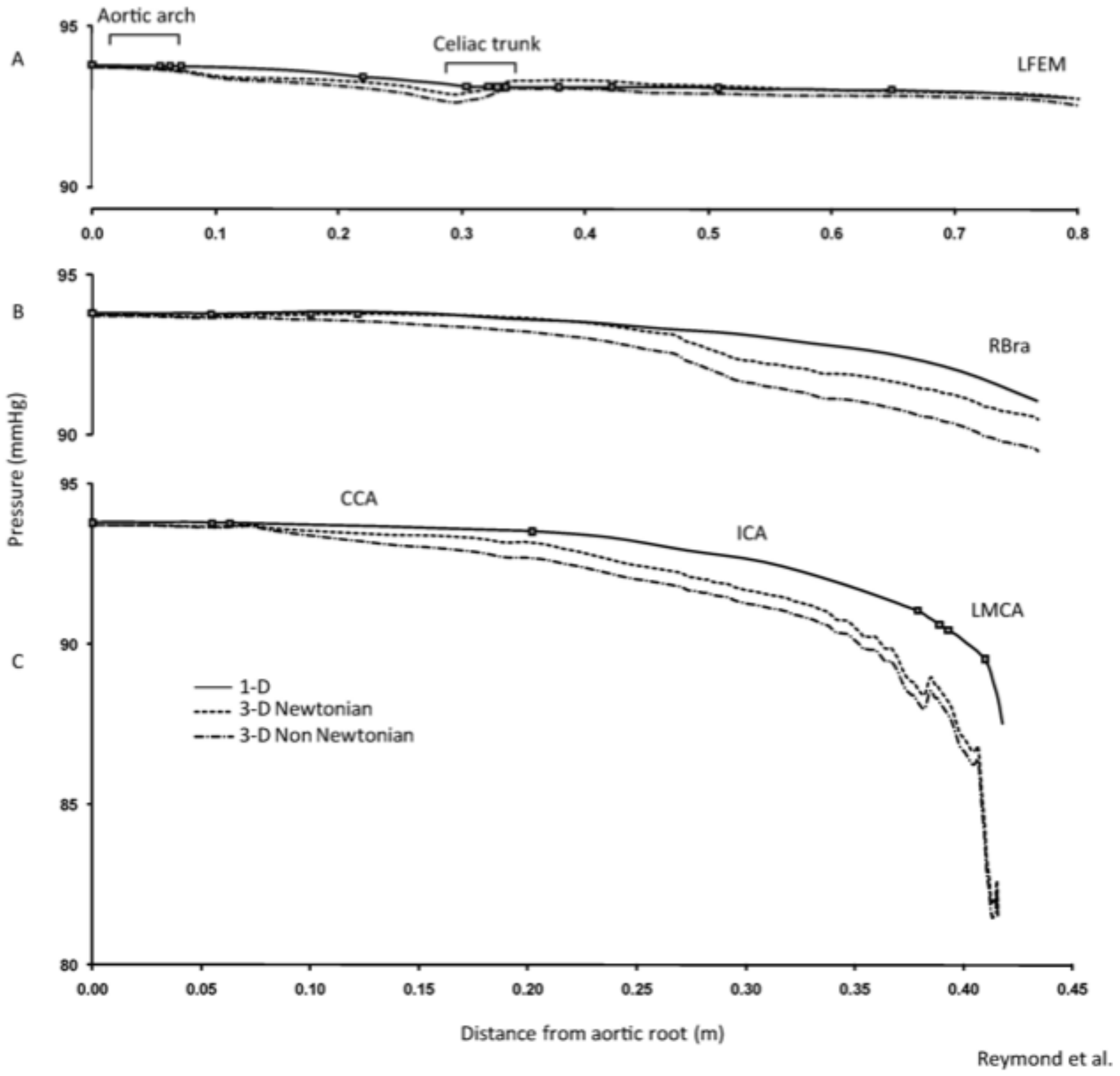


$$\frac{L_e}{D} = 0.064\,Re$$

INLET LENGTH. Flow entering a side branch results in skewed profile. It takes a certain inlet length before the velocity develops into a parabolic profile again.

3D and non-Newtonian effects



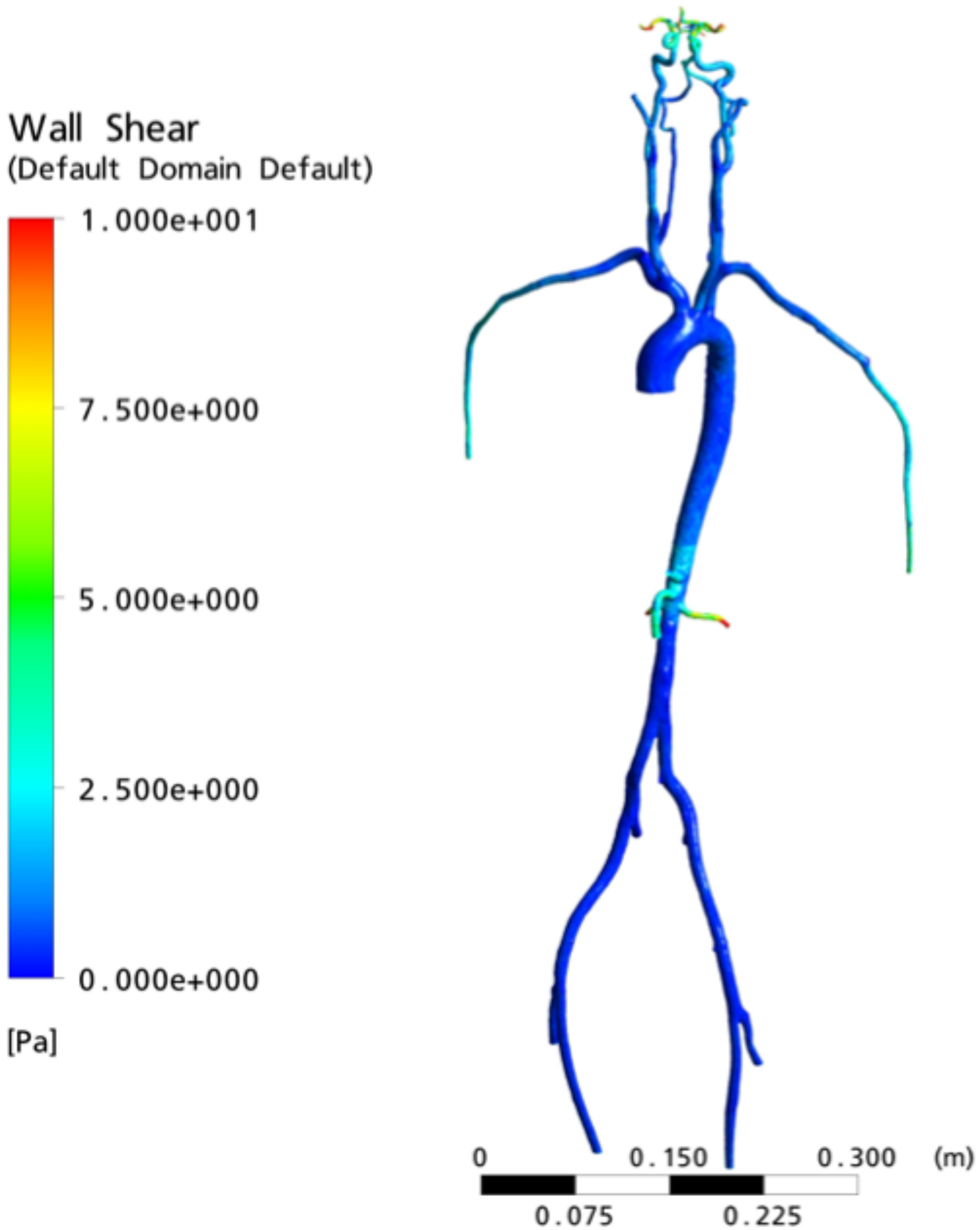


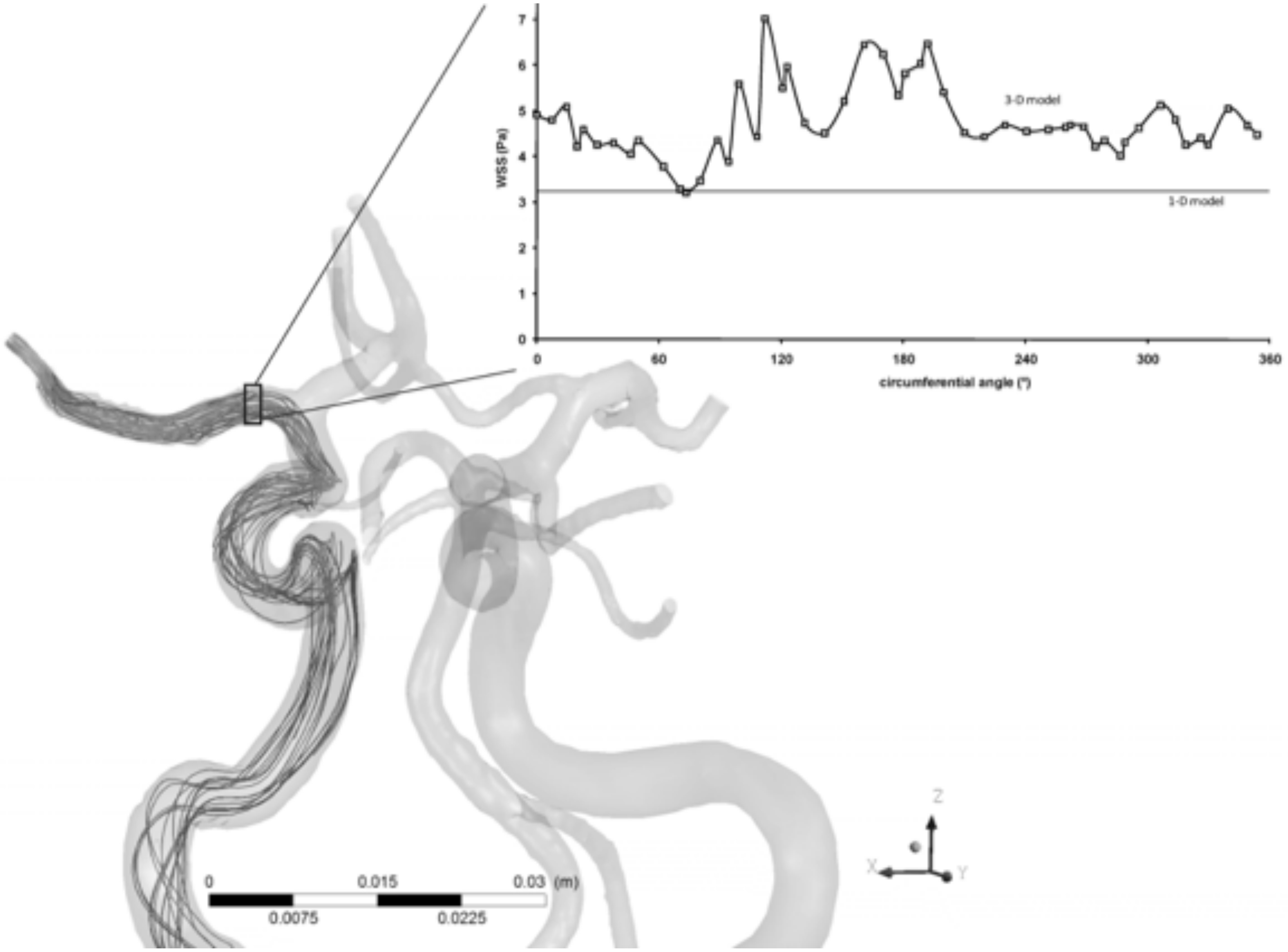
Remarks:

1) Non-Newtonian and 3D effects are negligible in the big vessels

2) Non-Newtonian and 3D effects are important in small peripheral parties and in particular in the tortuous vessels of the cerebral circulation

Shear stress distribution



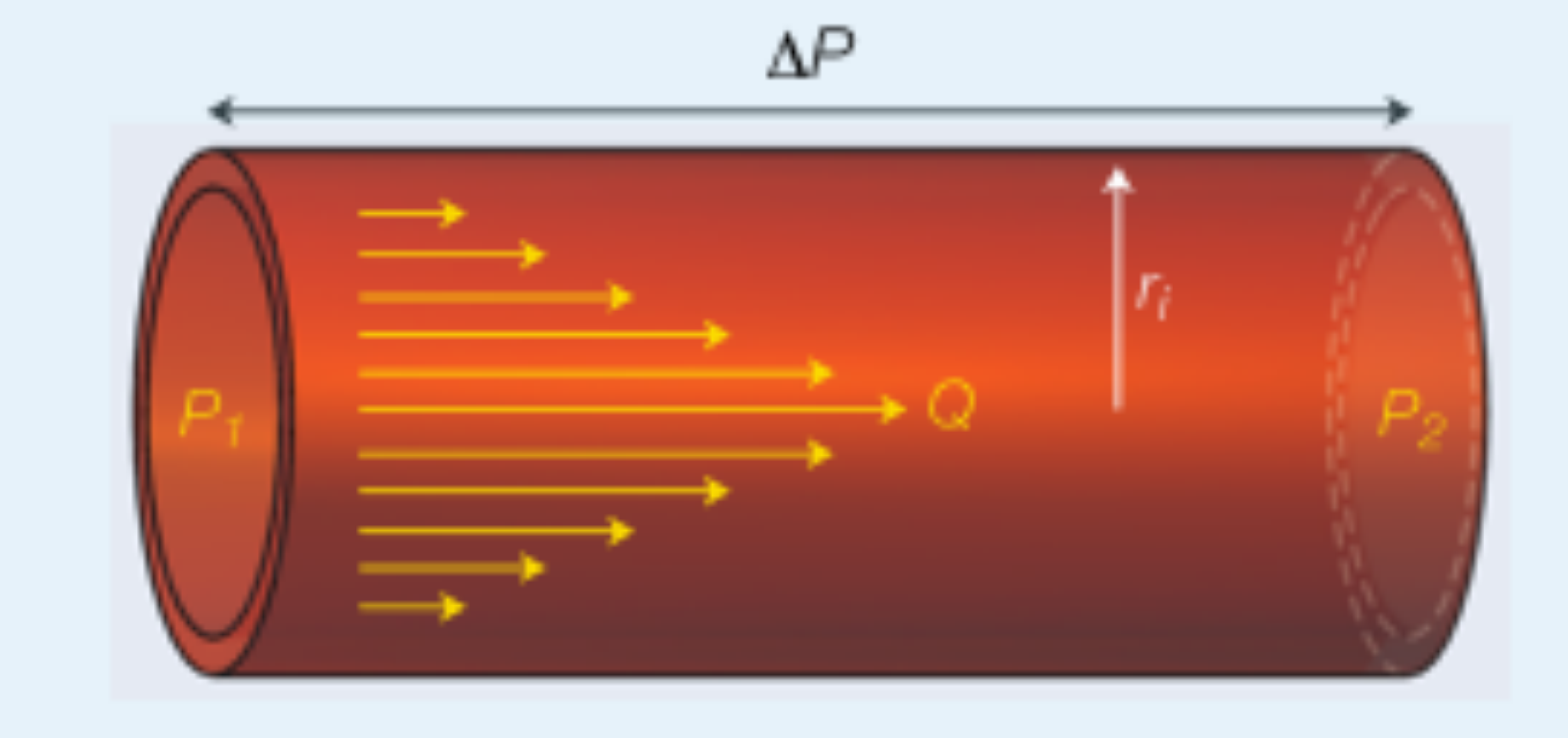


3D

Poiseuille (1D)

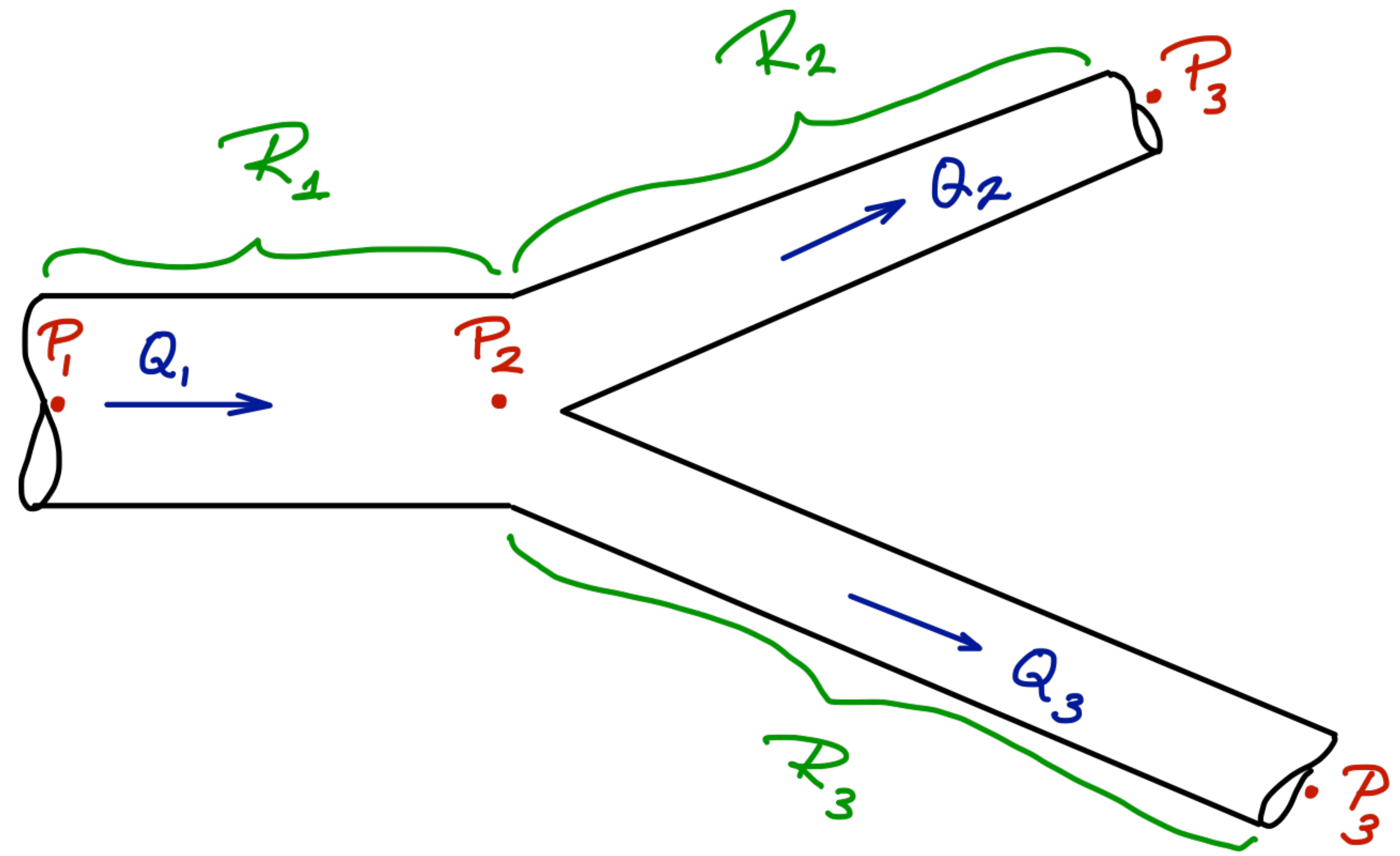
Conclusion: Wall shear stress in tortuous vessels is non-uniform and its value its underestimated when using the Poiseuille formula.

Resistance



$$R = \frac{\Delta P}{Q} = \frac{8\mu l}{\pi r^4}$$

Addition of resistances



Continuity of flow: $Q_1 = Q_2 + Q_3 = \frac{\Delta P_2}{R_2} + \frac{\Delta P_3}{R_3}$

$$\Rightarrow Q_1 = \frac{P_2 - P_3}{R_2} + \frac{P_2 - P_3}{R_3} = (P_2 - P_3)\left(\frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\Rightarrow \frac{Q_1}{P_2 - P_3} = \frac{1}{R_{2,3}} = \frac{1}{R_2} + \frac{1}{R_3}$$

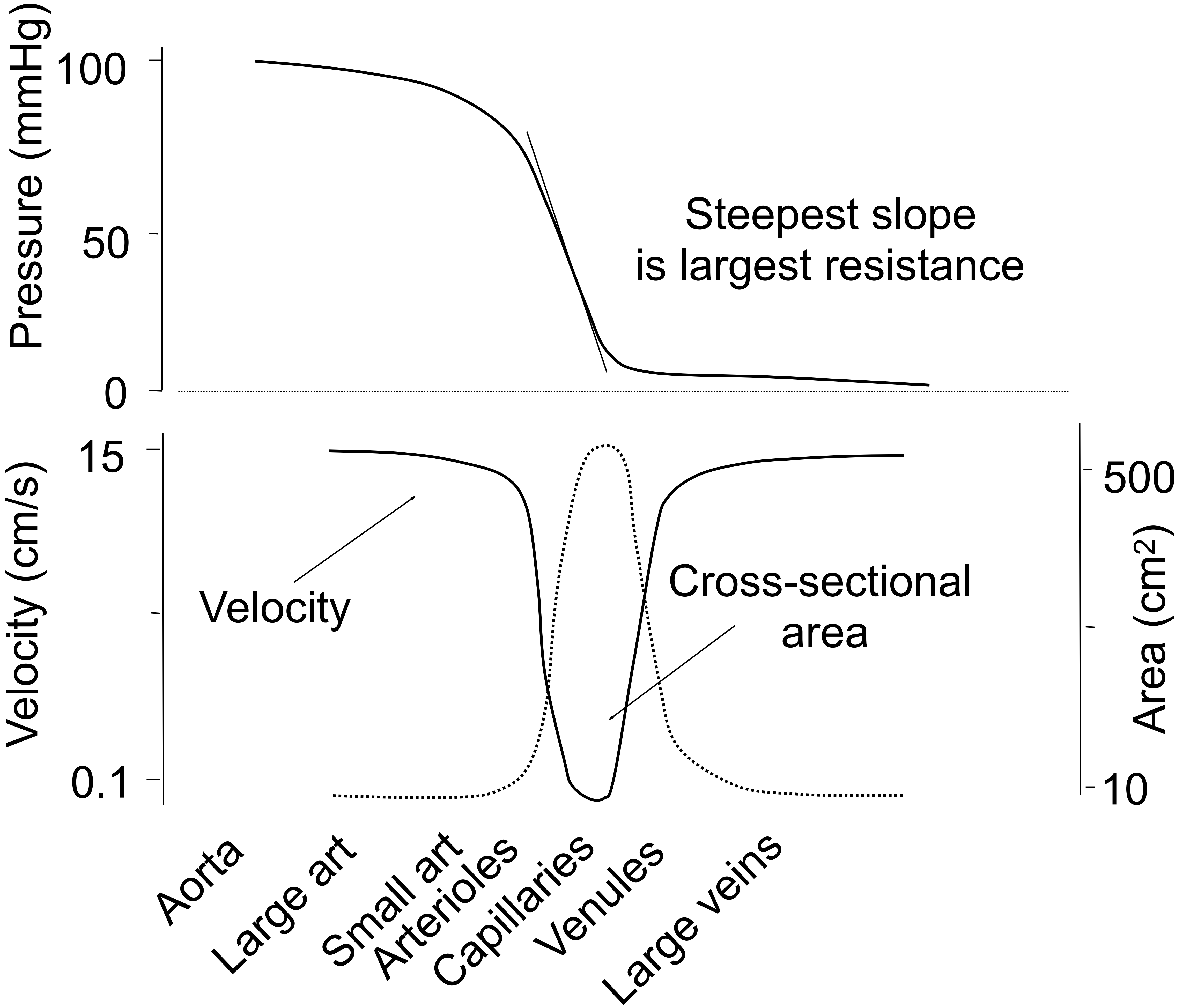
In parallel (addition of resistances)

$$R_{total} = \frac{P_1 - P_3}{Q_1} = \underbrace{\frac{P_1 - P_2}{Q_1}}_{R_1} + \underbrace{\frac{P_2 - P_3}{Q_1}}_{R_{2,3}}$$

$$\Rightarrow R_{total} = R_1 + R_{2,3}$$

In series

Distribution of resistances

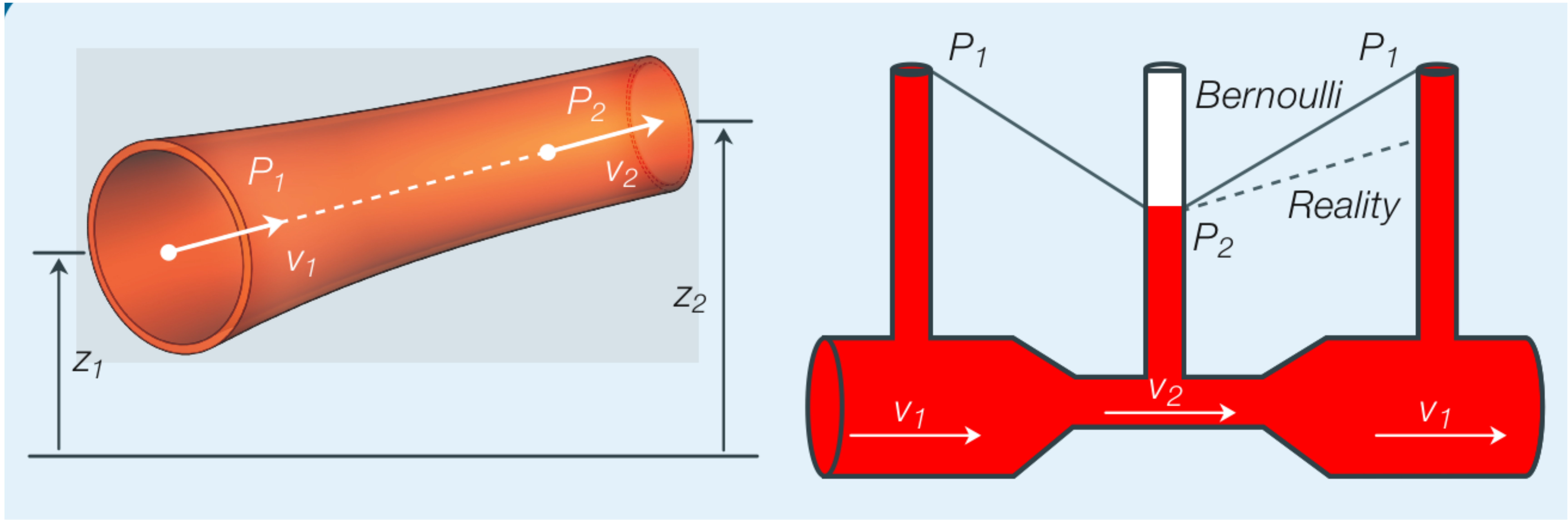


Compare resistances of the aorta, arterioles and capillaries

	Aorta	Arterioles	Capilaries
number of arterial segments in parallel (n)	1	$3x10^8$	$1.5x10^9$
Radius (r)	15 mm	7.5 μ	7.5 μ
Length (l)	50 cm	1 mm	1 mm
Resistance of single segment $R = \frac{8\mu l}{\pi r^4}$	$1.00x10^5$	$3.22x10^{15}$	$3.22x10^{15}$
Total resistance $R_{tot} = \frac{R}{n}$	$1.00x10^5$	$1.07x10^7$	$2.14x10^6$

The highest resistance is at the level of the arterioles

Bernoulli



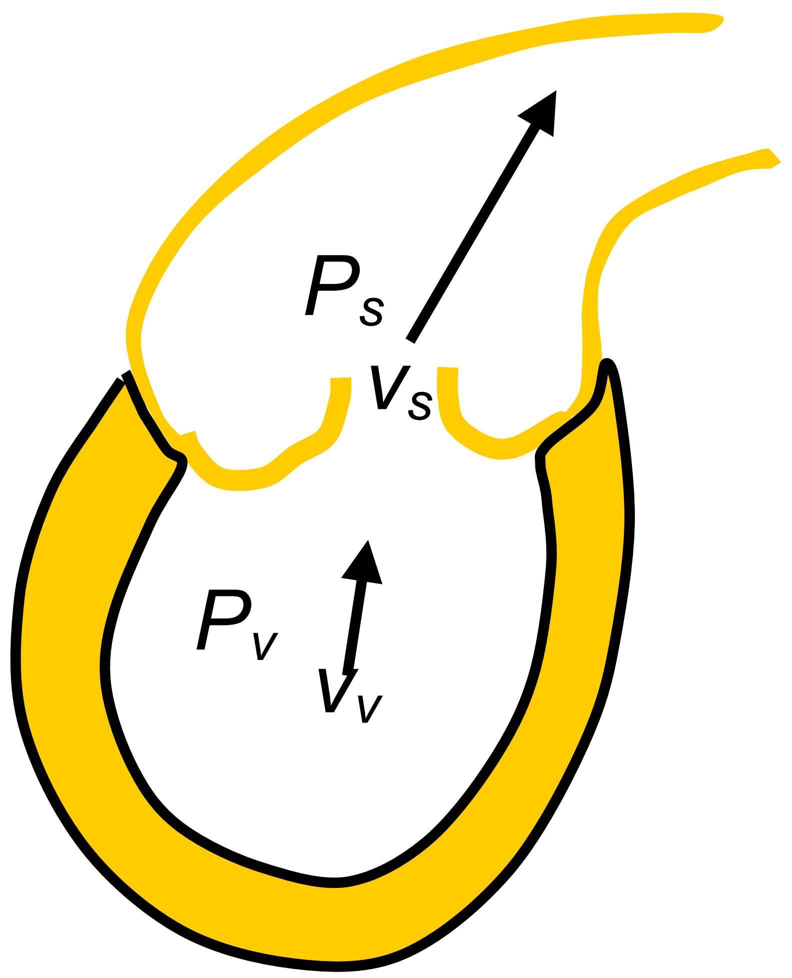
$$P_1 + 1/2\, \rho\, v_1^2 + \rho\, g\, z_1 = P_2 + 1/2\, \rho\, v_2^2 + \rho\, g\, z_2$$

$$\underbrace{p}_{\text{Pressure}} + \underbrace{\frac{1}{2}\rho V^2}_{\text{Dynamic Pressure}} + \underbrace{\rho g h}_{\text{Hydrostatic Pressure}} = ct$$

Conditions:

- No friction (no viscous losses)
- Steady flow
- Along a streamline

Example using the Bernoulli equation: the Gorlin equation for aortic stenosis



P_v: ventricular pressure
P_s: pressure at the aortic stenosis
V_v: velocity in ventricular lumen
V_s: velocity at the valvular stenosis

Let us consider flow through a stenosed valve according to Figure. Applying Bernoulli's law and assuming height differences can be neglected

$$P_v + \tfrac{1}{2}\rho V_v^2 = P_s + \tfrac{1}{2}\rho V_s^2$$

$$\Delta P = P_v - P_s = \tfrac{1}{2}\rho(V_s^2 - V_v^2)$$

The flow Q is the same at both locations, thus

$$A_v V_v = A_s V_s = Q$$

where A_v and A_s are the cross-sectional areas of ventricle and valve, respectively. Substituting this into the Bernoulli's equation we obtain:

$$\Delta P = P_v - P_s = \tfrac{1}{2}\,\rho Q^2(1/A_s^2 - 1/A_v^2)$$

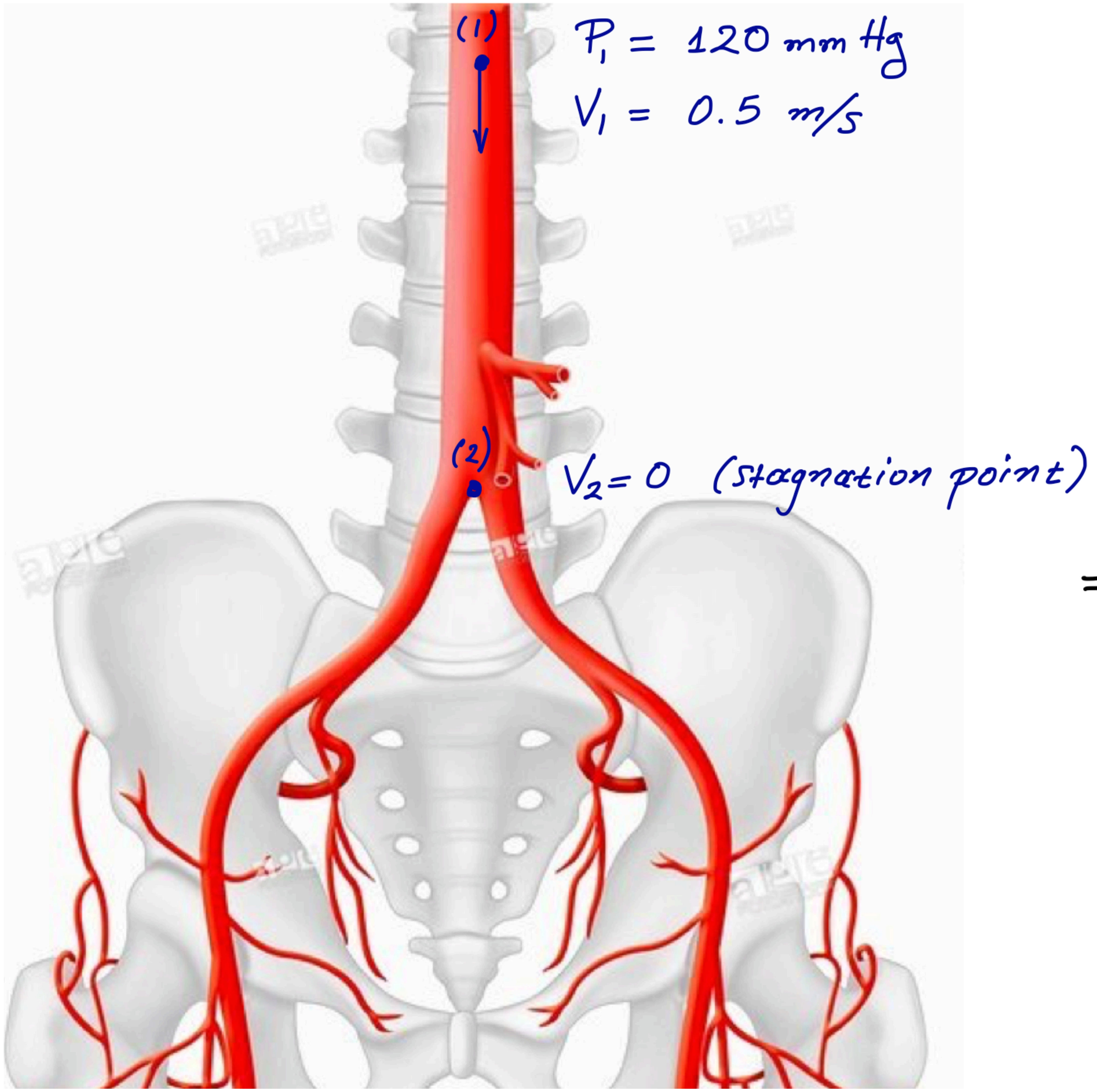
Since $A_s << A_v$ the equation can be simplified to:

$$\Delta P = \tfrac{1}{2}\rho Q^2 / A_s^2 = \tfrac{1}{2}\rho v_s^2$$

This relation has been used to **estimate effective area , A_s,** of a valvular stenosis by measuring flow and pressure gradient (e.g., using a pressure wire).

$$A_s = Q\sqrt{\frac{\rho}{2\Delta P}}$$

Example using the Bernoulli equation: pressure rise at the apex of the iliac bifurcation



At peak systole (P = 120 mmHg), the blood flowing in the lower abdominal aorta with a peak velocity $v \approx 0.5$ m/s hits the wall of the apex of the iliac bifurcation.

If it would come to a rest there, velocity is negligible ($v = 0$). On the basis of the Bernoulli equation

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2 \quad (V_2 \to 0)$$

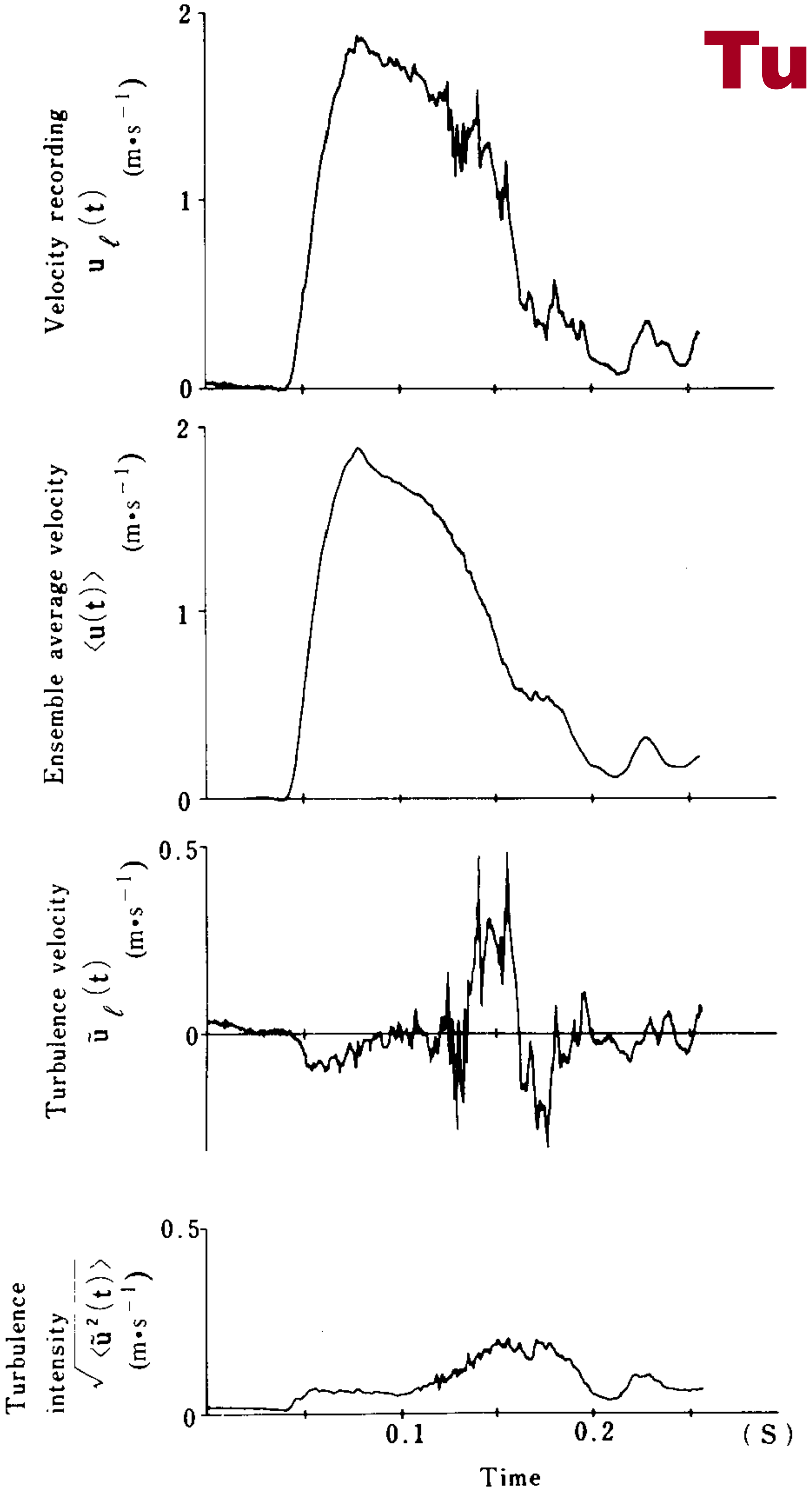
$$\Rightarrow P_2 = P_1 + \frac{1}{2}\rho V_1^2 = 120\ \text{mmHg} + \frac{1}{2} \cdot 1060\ \frac{\text{kg}}{\text{m}^3} \cdot 0.5^2\ \frac{\text{m}^2}{\text{s}^2}$$

$$= 120\ \text{mmHg} + 132\ \text{Pa} \cdot \left(\frac{1\ \text{mmHg}}{133\ \text{Pa}}\right)$$

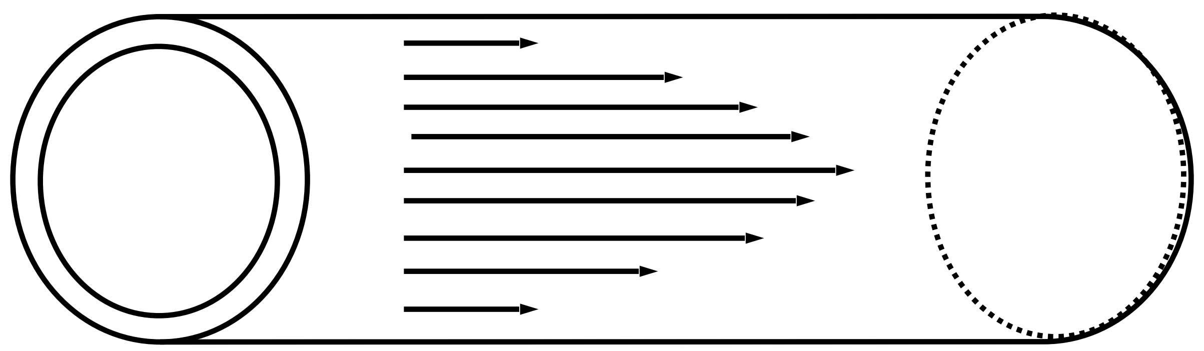
$$= 120\ \text{mmHg} + \underline{\underline{1\ \text{mmHg}}}$$

Pressure rise

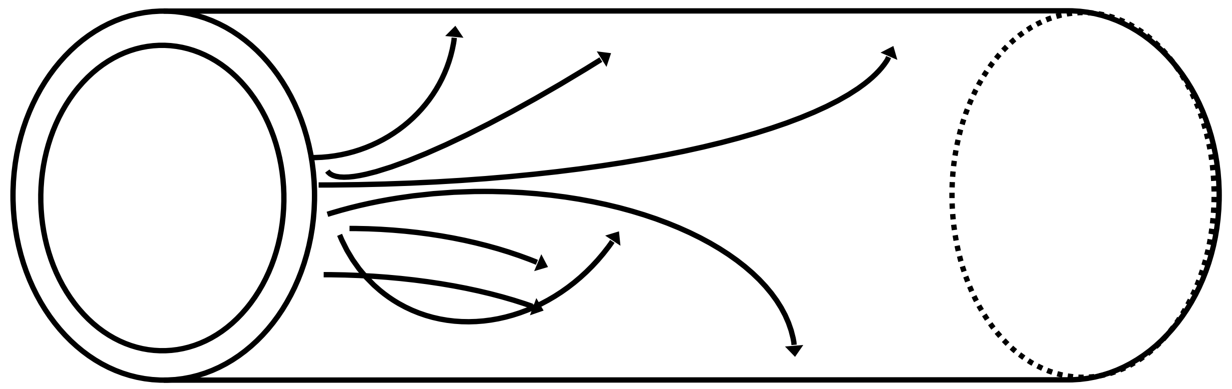
Turbulence



Laminar

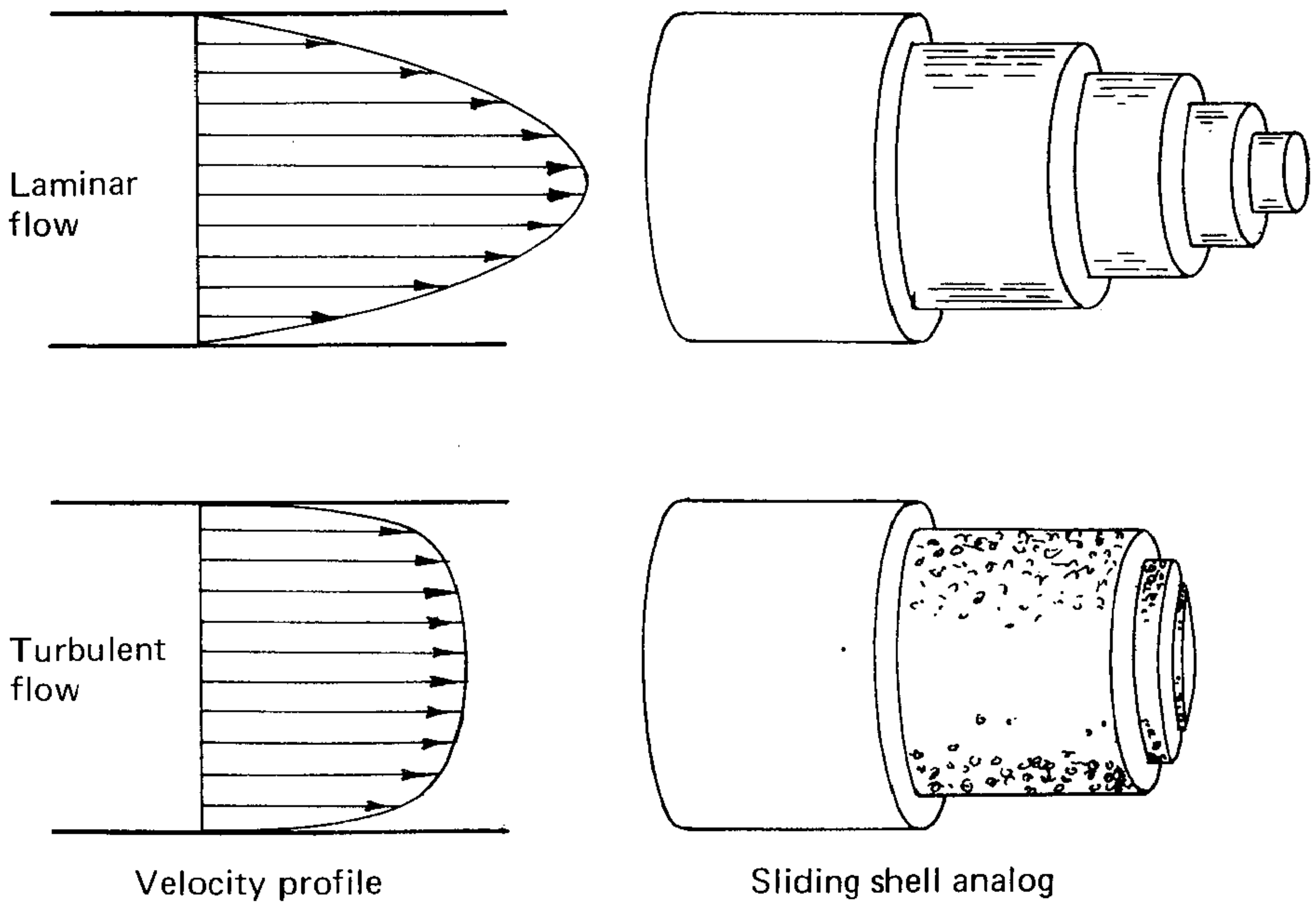


Turbulent

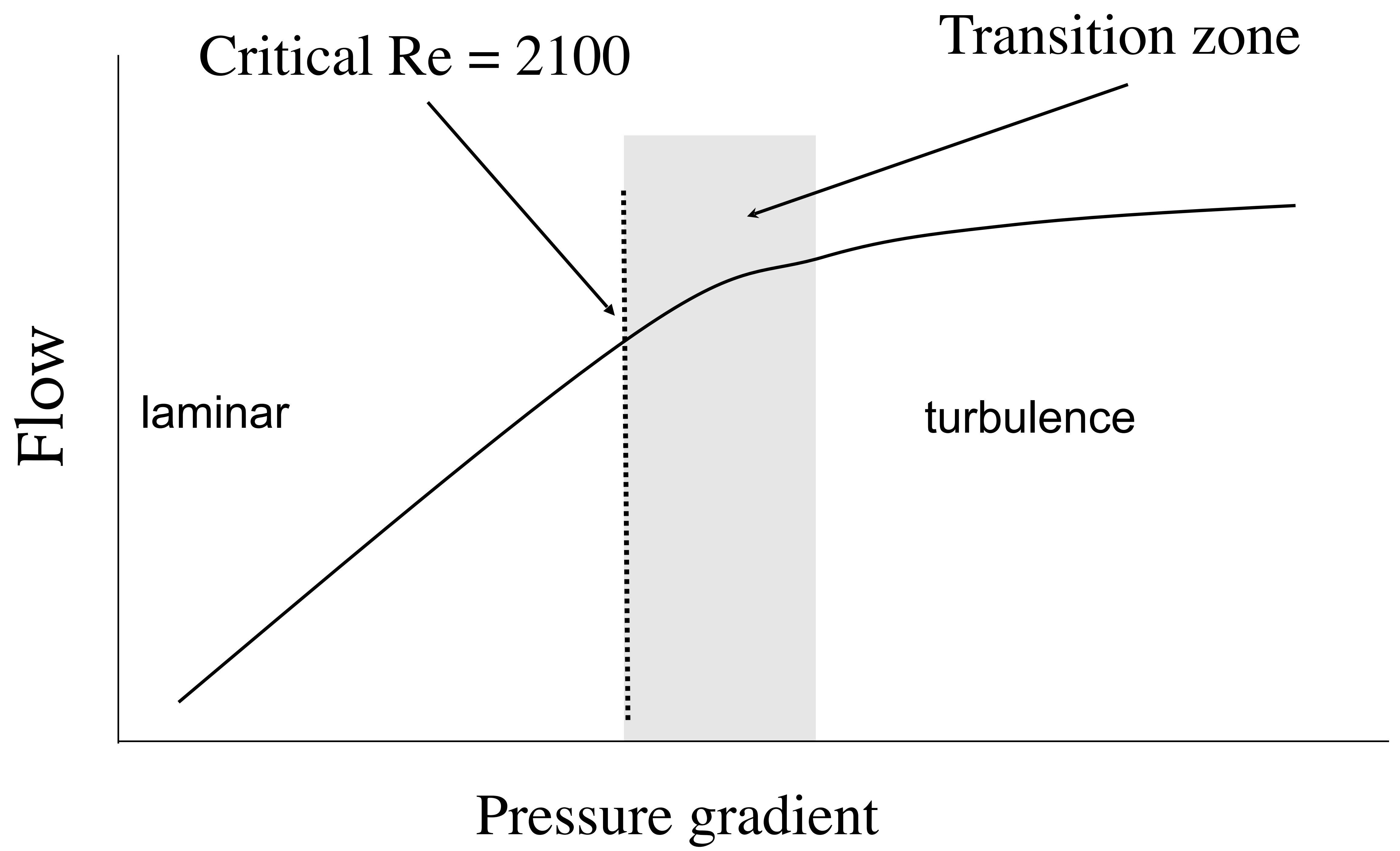


- “Random”, erratic movement of fluid particles
- Energy dissipation
- Increased friction:
  $\tau = \mu \cdot$ “shear rate”

Turbulence: velocity profile



Reynolds number and transition to turbulence



Reynolds number $\mathrm{Re} = \frac{\rho D V}{\mu}$ = $\frac{\text{Inertial effects}}{\text{Viscous effects}}$